Correspondence

Stability of an Asynchronous Swarm With Time-Dependent Communication Links

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***Abstract*—In this correspondence, we consider a simple model of N interacting agents with fixed or time-dependent communication links. We allow for asynchronous operation and time delays in the information flow. We show that the convergence of the states of the agents to a common value will be achieved, provided that old information is uniformly purged from the system. The considered model finds an application not only in swarming but also in other fields, including synchronization and distributed decision making or consensus seeking.**

***Index Terms*—Asynchronous, consensus, distributed agreement, dynamic neighborhood topology, multiagent systems, stability, swarms, synchronization.**

I. Introduction

Many social organisms aggregate in groups and have the ability to perform cooperative and coordinated behavior as a group: schools of fish or herds of animals can orient their motion in the same direction, and thousands of fireflies can synchronize their flashing. Such a spectacular behavior is achieved in a distributed manner despite the fact that each individual in the group has access to only local information. Principles developed from studying such naturally distributed systems could be very useful in characterizing and analyzing mechanisms for coordination and control of networked control systems and multiagent systems such as groups of unmanned aerial vehicles or autonomous robots.

The information flow constraints can have a considerable impact on the system stability. Recent years have witnessed significant research focused on the effect of information flow on systems composed of multiple interacting entities (agents). Some examples include [1]–[4]. Jadbabaie *et al.* [1] considered a simple model of n interacting particles with time-dependent bidirectional communication links. They showed that the 1-D system state (the heading in their case) will converge to the same value, provided that the union of the communication graphs is uniformly jointly connected. Ren and Beard [4] extended the results to unidirectional communication and relaxed the connectivity assumption to the assumption that the union of the communication graphs has a spanning tree. Independently, Moreau [2] considered a more general nonlinear interaction model and showed that, under unidirectional communication, consensus will be achieved if, for any uniformly bounded time interval, there is an agent that is connected to all other agents (equivalent to the spanning tree assumption of [4]), whereas, for bidirectional communication, the same will be achieved without uniformity in connectedness. Later, in [3], the same author relaxed the uniformity in the connectedness assumption for the unidirectional case as well but assumed uniformity in the communication cycles in the graph. The results in [2] and [3] are based on convexity analysis and are more general than those in both [1] and [4]. Another relevant reference is the work in [5], where the authors consider a group of unicycles and show that the rendezvous problem is solvable (i.e., a controller that achieves stabilization to a point exists) if and only if the communication graph has a globally reachable node or basically a spanning tree (in contrast to the work in [1]–[4], where the analysis is based on a specific control law). However, they consider only the fixed-communication-topology case.

Other relevant articles include [6]–[8]. Fax and Murray [6] emphasize the role of information flow and graph Laplacians and derive Nyquist-like criterion for stabilizing vehicle formations. Olfati-Saber and Murray [7] describe consensus protocols for networks of dynamic agents with fixed and switching communication topologies and show that the connectivity of the network is key in reaching consensus. They determine a class of directed communication graphs, which guarantee reaching average consensus, and they establish connection with the Fiedler eigenvalue of the graph Laplacian and the speed of convergence. Moreover, they also consider time delays and channel filtering affects. Sepulchre *et al.* [8] study connections between models of coupled phase oscillators and kinematic models of swarms (groups of self-propelled particles) and design control laws for stabilizing collective motions of groups.

Most of the aforementioned papers (with the exception of [7], which considers bounded and equal time delays in continuous time) and, in particular, [1]–[4] consider synchronous motion and perfect information. In other words, all the agents move simultaneously and every agent knows the exact state of its neighbors (i.e., the agents with which it can currently communicate). However, distributed multiagent systems inherently operate in an asynchronous manner with possible imperfect information due to delays in communication and/or sensing. Moreover, these properties can have a considerable impact on the system stability as well. Therefore, any realistic model of such systems should possess these properties.

Asynchronous models for modeling swarms have also been considered in the literature [9]–[11]. Liu *et al.* [9] consider 1-D discrete-time asynchronous models for both stationary and mobile swarms and prove asymptotic convergence under total asynchronism conditions and finite-time convergence under partial asynchronism conditions (i.e., total asynchronism with a bound on the maximum possible time delay). For the mobile swarm case, they prove that cohesion will be preserved during motion under conditions expressed as bounds on the maximum possible time delay. In [10], the work in [9] has been extended to the multidimensional case by imposing special constraints on the "leader" movements and using a specific communication topology. In [11], we considered a similar model to the one in [9]. In particular, we used the representation of a single swarm member as in [9] and a different mathematical model for the interagent interactions and motions in the swarm as well as different tools for analysis. Other recent references dealing with asynchronous algorithms for cooperative coordination and control include [12]–[15]. The work in [12] deals with optimal distribution (or coverage) of the agents in a region, whereas those in [13]–[15] deal with the rendezvous [13] or the gathering (i.e., rendezvous in finite time) problem [14], [15]. Beni [16] showed that asynchronous swarms may exhibit "richer behavior" compared to their synchronous counterparts. In particular,

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he determined that asynchronous systems of swarms converge to the same fixed points as their synchronous counterparts, and moreover, the asynchronous systems may reach fixed points that are unreachable for the synchronous ones. In a recent work, we empirically investigated the effects of asynchronism, time delays, and neighborhood size for systems of self-propelled particles [17] and analyzed the stability of an asynchronous system under cyclic pursuit [18].

In this correspondence, we present a model of an asynchronous system of N interacting agents via fixed or dynamic communication/interaction links. The model also includes possible time delays (which could be due to delays in communication or sensing). We show that convergence to a common state (or consensus) will be achieved, provided that old information is uniformly purged from the system. We employ techniques from [19] to prove the stability of an asynchronous interacting swarm in an n-dimensional space. The model extends the work in [1]–[4] and finds application not only in swarming but also in other fields, including synchronization and distributed decision making or consensus seeking. Although the setup of the problem here has similarities to the ones in [9]–[11] (all are based on [19]), it also has important differences: 1) in [9]–[11], a specific leader–follower-based communication topology is specified *a priori*, and the results are derived based on that topology only; on the contrary, here, a leaderless swarm is considered, and only a connectedness property of the communication topology is specified; 2) as a difference from [9]–[11], we also allow for a dynamically changing communication topology.

Recently, Angeli and Bliman [20] provide an extension of the result by Moreau [2] by relaxing the convexity assumption and allowing for a known and bounded time delay. The results here have been independently obtained and are also different from those in [20]. In particular, our system operates in an asynchronous manner, whereas the system considered in [20] is synchronous. Moreover, the analysis here is based on different mathematical tools. Other more recent works similar to this correspondence are [21] and [22]. Fang *et al.* [21] summarize the recent results on synchronous consensus protocols, briefly discusses asynchronous protocols, poses some open questions, and shows some simulation-based preliminary results on asynchronous protocols using a custom Java-based simulator. No results in the form of this correspondence are presented. Aside from discussing the current results in the literature, reference [22] presents some new results for systems/protocols with delays as well. Asynchronous motion is not considered in [22]. The authors also claim that most of the recent results obtained in the literature (e.g., those in [1]–[4]) are special cases or could be approached from the point of their earlier work on parallel and distributed computation. They point out the power of the asynchronous convergence theorem in [19] and state that it can be used or modified accordingly to address many of the problems/cases considered in the literature. In this correspondence, we take that exact approach (i.e., use the results in [19]) to prove the convergence for the asynchronous case with time delays. (We also noticed the usefulness of the results in [19] during our studies on 1-D asynchronous swarms [11].) We would like to stress that the results in this correspondence have been independently obtained and before the works in [21] and [22]. A preliminary version of the current correspondence was presented in the Systems and Control Theory Workshop in Gebze, Kocaeli [23].

II. Swarm Model

Consider a multiagent system (a swarm) consisting of N individuals with states denoted by $x^i \in \mathbb{R}^n$, which could be position, orientation, synchronization frequency, or some other physical variable depending on the problem. It could also represent some other information (e.g., cognitive variables) to be distributively agreed upon by the agents. Assume that each agent can communicate only with a fixed or time-dependent (i.e., dynamic) subset of the swarm called its neighbors. In applications, this subset may be determined based on the distance between the agents (due to, for example, the finite range of communication or sensing), based on the physical layout or topology of the environment (walls may cause agents to be out of sight, etc.), or by some other (e.g., heuristic, probabilistic, or ad hoc) means. Given agent i, we denote with $S_i(t)$ the set of its neighbors and with $N_i(t)$ the number of its neighbors at time t. In other words, $N_i(t)$ denotes the number of elements in set $S_i(t)$. We assume that each agent updates its state by

$$x^i(t+1) = \frac{1}{w_i(t)} \left[w_{ii}(t)x^i(t) + \sum_{j \in S_i(t)} w_{ij}(t)x^j\left(\tau_j^i(t)\right) \right] \qquad \forall t \in T^i \quad (1)$$

where $x^i(t)$, $i = 1, \ldots, N$, represents the state of member i at time t, the variables $w_{ij}(t)$, $1 \leq i, j \leq N$, are the weighting factors, and $w_i(t) = w_{ii}(t) + \sum_{j \in S_i(t)} w_{ij}(t)$, $i = 1, \ldots, N$. The set $T^i \subseteq T = \{0, 1, 2, \ldots\}$ is the set of time indexes at which member i updates its state. At the other time instants, member i is stationary, i.e.,

$$x^i(t+1) = x^i(t) \qquad \forall t \notin T^i. \quad (2)$$

It is assumed that the weighting factors satisfy $w_{\min} \leq w_{ij}(t) \leq w_{\max}$ for some $w_{\min} > 0$ and $w_{\max} < \infty$ for all i, j, and all t. The variables $\tau_j^i(t)$, $j \in S_i(t)$, $i = 1, \ldots, N$, are used to represent the time index of the state information of $j \in S_i(t)$ to which member i has access to. They satisfy $0 \leq \tau_j^i(t) \leq t$ for $t \in T^i$, where $\tau_j^i(t) = 0$ means that member i has not yet obtained any information about member j (it still has the initial state information), whereas $\tau_j^i(t) = t$ means that it has the current state information of member j. The difference $(t - \tau_j^i(t)) \geq 0$ can be viewed as a sensing delay or a communication delay in obtaining information about agent j by agent i. Note that this definition can represent both of the following cases: 1) The agents are memoryless, and when, at time t, agent j becomes a neighbor of agent i, it performs (relative) position sensing of agent j but gets an old distance due to the time delay in sensing. 2) The agents have memory and keep record of all its past neighbors, and when, at time t, agent j becomes a neighbor of agent i, it may perform sensing of the state of agent j with some probability or uses the old recorded information about it. The first case may arise in, for example, robot gathering algorithms such as those considered in [13]–[15], whereas the second may serve as a crude representation of interactions in social networks (and adaptation of attitudes or beliefs, for example). (See [24] for more information on social networks.)

Note that $j \in S_i(t)$ does not necessarily mean that $i \in S_j(t)$. In other words, we assume unidirectional communication. Moreover, even if $j \in S_i(t)$ and $i \in S_j(t)$ simultaneously hold, this does not imply that $\tau_j^i(t) = \tau_i^j(t)$. In other words, even if two members i and j are mutually neighbors of each other at a given time instant, it does not mean that they have the current or equally outdated information about each other, implying that they do not necessarily communicate information to each other simultaneously. In fact, it is assumed that they can obtain information about each other, as well as update their states at totally independent time instants.

The elements of set T (and therefore of T^i) should be viewed not as actual times but as indexes of the sequence of ordered physical times $\mathcal{T} = \{t_0, t_1, t_2, \ldots\}$, where $t_i < t_{i+1}$, at which the updates generated by all the agents occur (similar to the times of events in discrete-event systems). In other words, the elements of set T are integers that can be mapped into the actual times (i.e., the times of the events) $\{t_i | t_i < t_{i+1}\}$, and the physical time intervals $(t_{i+1} - t_i)$ between subsequent indexes (events) are not necessarily uniform. Sets T^i are independent from each other for different i. However, it is possible to have $T^i \cap T^j \neq \emptyset$ for $i \neq j$ (i.e., it may happen that, sometimes, two or more members simultaneously update their states). Note that the set T is only needed for analysis purposes, and in order to implement the iteration in (1), it is not required for the agents to know it. Similarly, the agents do not need to know neither the sets T^i nor the set of physical times $\mathcal{T}$. One can view these sets as the global times viewed/observed by an external observer, while the agents operate on their local independent clocks. In other words, there is no need for a global clock or a means for synchronization for the implementation of equation (1).

We would like to emphasize here that the model in (1) is also suitable for applications in which the relative positions/states can be measured, instead of actual positions/states. To see this, note that, by taking $w_{ij}(t) = 1$, $1 \leq i, j \leq N$, and rearranging, (1) can be written as

$$x^i(t+1) = x^i(t) + \frac{1}{N_i(t)+1} \times \sum_{j \in S_i(t)} \left[x^j\left(\tau_j^i(t)\right) - x^i(t)\right] \qquad \forall t \in T^i.$$

In addition, note that (1) can be used for a high-level representation of the rendezvous or gathering problem in asynchronous multivehicle systems, such as those considered in [13]–[15]. In particular, consider a networked system of agents that have continuous-time vehicle dynamics and have sensing/communication, computation and motion capabilities. Assume that they operate on the (infinite) sequence of behaviors *wait–sense–compute–move*. Let agent i be located at $x^i(t)$ at time index t. After performing a sensing of its current neighbors, during its *compute* state/behavior, it computes using (1) the new *waypoint* (*position point* or *path point*) $x^i(t+1)$. Then, it moves (using some local control) in some (unspecified but finite) amount of time to its new position (the *move* behavior), waits for some amount of time, and performs new neighbor position sensing. Note that, during the *wait*, *sense*, and *move* behaviors, no new waypoints are computed, which corresponds to (2). The agent does not need to know at which time instant the other agents move, and there is no need for a global clock. The asynchronism in the system comes from the fact that the times for the completion of the behaviors and, in particular, of the *move* behavior is not necessarily uniform. The model in (1) provides a high-level view of such systems, and the results developed in this correspondence will hold for such systems as well. However, here, we are not concerned with the low-level vehicle dynamics and control of the agents. The only requirement is that they should be designed such that the agents move to their next computed waypoints in some finite (unspecified) amount of time.

The swarm model in (1) is different from the ones considered in [1]–[4] in two main aspects: 1) the agents update their states in asynchronous manner and 2) they do not necessarily have the exact information about the states of the other agents. Therefore, we believe that it is more suitable for describing the operation of distributed multiagent systems since these features are natural properties of such systems. Synchronous operation is difficult to implement even in artificial multiagent systems such as swarms of robots since it requires a global clock to which all the agents must be subjected to, undermining the distributed (decentralized) nature of the problem.

In [1]–[4], it was shown that all the states converge to a common value, i.e., as $t \to \infty$, we have

$$\lim_{t \to \infty} x^i(t) = x_c \tag{3}$$

for some constant vector $x_c \in \mathbb{R}^n$ and for all i. (Actually, in [1], [3], and [4], 1-D state $x^i \in \mathbb{R}$ was considered, but the results hold for $x^i \in \mathbb{R}^n$ as well.) The question is whether the same will be achieved here despite the asynchronism and the time delays.

We will use a directed graph to represent the interaction (information flow) topology. Let $\mathcal{G}(t) = (\mathcal{N}, \mathcal{A}(t))$ denote the information flow or interaction graph of the system at time t, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is the fixed set of nodes and $\mathcal{A}(t) \subset \mathcal{N} \times \mathcal{N}$ denotes the set of directed arcs (or information flow links) at time t. Agent $i \in \mathcal{N}$ denotes the ith node or vertex of the graph, whereas the arc $(i, j) \in \mathcal{A}(t)$ represents a directed information flow link from agent i to agent j at time t. In other words, if $(i, j) \in \mathcal{A}(t)$, then agent j can receive or obtain information from agent i at instant t, implying that $i \in S_j(t)$. Note once more that the information flow is unidirectional, meaning that $(i, j) \in \mathcal{A}(t)$ does not imply that $(j, i) \in \mathcal{A}(t)$.

Agent i is said to be connected to agent j if there is a directed path from i to j. In other words, there is a sequence of arcs (i_1, i_2), $(i_2, i_3), \ldots, (i_{p-1}, i_p)$, such that $i = i_1$ and $j = i_p$. A directed tree is a directed graph in which every node, except the root, has exactly one incoming edge (arc). If the tree connects all the vertices of the graph, then it is called a spanning tree. Note that, if a graph has a spanning tree, then there is at least one agent that is connected to all the other agents.

In this correspondence, we assume that the communication topology can be time dependent. As in [4], denote the set of all possible interaction graphs as $\bar{\mathcal{G}} = \{\mathcal{G}_1, \ldots, \mathcal{G}_M\} = \{\mathcal{G}_p | p = 1, \ldots, M\}$. Note that $\bar{\mathcal{G}}$ is finite, and for each t, we have $\mathcal{G}(t) \in \bar{\mathcal{G}}$. The union of a set of graphs $\{\mathcal{G}_i = (\mathcal{N}, \mathcal{A}_i)\} \subset \bar{\mathcal{G}}$ with the same vertex set is the graph defined as $\cup \mathcal{G}_i = (\mathcal{N}, \cup \mathcal{A}_i)$. We say that a sequence of graphs $\{\mathcal{G}(t)\}$ has a spanning tree over an interval $\mathcal{I}$ if the graph $\cup_{t \in \mathcal{I}} \mathcal{G}(t)$ has a spanning tree. Let $\mathcal{P} = \{1, \ldots, M\}$ and σ: $T \to \mathcal{P}$ denote the switching sequence (of the communication graphs). In addition, given a switching sequence $\sigma(t)$, denote the corresponding sequence of communication graphs as $\{\mathcal{G}_{\sigma(t)}\} = \{\mathcal{G}_p(t) = (\mathcal{N}, \mathcal{A}_p(t))\}$. In this correspondence, we will build on the results in [1]–[4]. Therefore, we have the following assumption:

Assumption 1: The switching sequence $\sigma(t)$ is such that there exists a constant $I \geq 0$ such that, for every interval $\mathcal{I}$ of length I, the corresponding sequence of communication graphs $\{\mathcal{G}_{\sigma(t)}\}$ has a spanning tree.

In the case of synchronous motion and perfect information, Assumption 1 is sufficient for achieving consensus [2], [4]. However, in the case of asynchronism and time delays, there is a need for extra conditions imposed on the information updates or the delay time.

In the next section, we briefly discuss the system under total synchronism and set up the stage for the main result of this correspondence: the convergence of the asynchronous system in (1) and (2).

III. System Under Total Synchronism

We start with the following assumption.

Assumption 2: (Synchronism, No Delays): Sets T^i and times $\tau_j^i(t)$ satisfy $T^i = T$ for all i and $\tau_j^i(t) = t$ for all i and $j \in S_i(t)$.

This assumption states that all the swarm members will move at the same time instants. Moreover, every member will always have the current state information of its neighbors. Under this assumption, the motion dynamics of the system become

$$x^i(t+1) = \frac{1}{w_i(t)} \left[w_{ii}(t) x^i(t) + \sum_{j \in S_i(t)} w_{ij}(t) x^j(t) \right] \tag{4}$$

for all $t \in T$ and all i.

It was shown in [2]–[4] that, under Assumption 1, for the synchronous system with the dynamics in (4), condition (3) is satisfied. Here, we will use this result to prove the convergence for the asynchronous system. First, we briefly introduce the notation that will be used in the next section.

Define set $X(t) \subset \mathbb{R}^n$ as

$$X(t) = \{x \in \mathbb{R}^n | m(t) \leq x \leq M(t)\}$$

where

$$m(t) = \min_{i=1,\ldots,N} \{x^i(t)\}$$
$$M(t) = \max_{i=1,\ldots,N} \{x^i(t)\}.$$

Here, $m(t)$ and $M(t)$ are vectors of dimension n, and the inequality sign and the min and max operators are operated elementwise. We would like to emphasize that the values of $m(t)$ and $M(t)$ and, therefore, the sets $X(t)$ depend on the initial configuration $x(0)$ of the system as well as the switching sequence $\sigma(t)$.

Note that $\{m(t)\}$ is nondecreasing and $\{M(t)\}$ is nonincreasing along the solutions of (4). In other words, we have $m(t+1) \geq m(t)$ and $M(t+1) \leq M(t)$ for all t, implying that $X(t+1) \subseteq X(t)$ for all t. This is because of the convexity of the weighted averaging in (4). By taking a convex combination between a set of numbers/points, the minimum value cannot decrease, and the maximum value cannot increase. Therefore, the mapping consisting of the concatenation of all (4) for all i is a contraction mapping [25] in $\mathbb{R}^{N \times n}$. These were shown (or implied) in [2]–[4]. Since $m(t)$ and $M(t)$ are bounded, their limits exist, and from the results in [2]–[4], we know that they are equal. In other words, we have

$$\lim_{t \to \infty} m(t) = m = M = \lim_{t \to \infty} M(t).$$

This final value is the one defined as x_c in (3). Note that it also depends on $x(0)$ and $\sigma(t)$.

In addition to the fact that $m(t)$ is nondecreasing and $M(t)$ is nonincreasing it is guaranteed that an increase in $m(t)$ or a decrease in $M(t)$ will occur in a few time steps, the number of which is bounded by the maximal information flow path in the communication graph (which is always less than N) and the uniformity in connectivity parameter I in Assumption 1. Then, it is guaranteed that, for all t, either $m(t+IN) > m(t)$ or $M(t+IN) < M(t)$, which implies that $X(t+IN) \subset X(t)$. Note that, in practice, based on the switching sequence $\sigma(t)$, usually it may happen that $X(t+\eta) \subset X(t)$ for some $\eta < IN$, and IN is the worst case interval for which the preceding condition is guaranteed for any switching sequence satisfying Assumption 1.

Let $Y(k) = X(kIN)$. Then, $Y(k+1) \subset Y(k)$. In addition, define

$$\bar{Y}(k) = \underbrace{Y(k) \times Y(k) \times \cdots \times Y(k)}_{N \text{ copies of } Y(k)}$$

and let $Y_c = x_c \times x_c \times \cdots \times x_c$. Then, from the preceding discussion, we have

$$Y_c \subset \cdots \subset \bar{Y}(k+1) \subset \bar{Y}(k) \subset \cdots \subset \bar{Y}(0) \subset \mathbb{R}^{N \times n}.$$

In other words, since (3) holds, under the dynamics in (4), the state of the system converges to Y_c (for some x^c). In particular, any sequence $\{y_k\}$ such that $y_k \in \bar{Y}(k)$ converges to Y_c. These properties will be very useful in the proof of the asynchronous system, to which we return in the next section.

IV. Main Result

In this section, we return to the asynchronous system in (1) and (2). We start with an assumption that allows the members to move at totally independent time instants. However, it also guarantees that the members will perform measurement/communication with their neighbors and a state update/move in, at most, B time steps for some finite B. In other words, there is uniformity in the measurement/communication as well as the update/move times, or basically, the time delay and the times between two moves are uniformly bounded. Note that the value of bound B does not need to be known by the agents. It is needed for analysis purposes, and it is sufficient for it to exist. The analysis here is based on the work on the parallel and distributed computation in [19].

Assumption 3: There exists a finite positive constant B such that, for every agent i and for all $t \geq 0$, two conditions hold.

1) At least one of the elements of $\{t, t+1, \ldots, t+B-1\}$ belongs to T^i.
2) Given the switching sequence $\sigma(t)$, for every $j \in S_i(t)$, we have $t - B < \tau_j^i(t) \leq t$.

Note that the preceding assumption is a very reasonable assumption. Basically, it states that any agent performs a move in, at most, B time steps and that the information about the neighbors (used by the agent during the determination of its next state/waypoint) is outdated by, at most, B time steps, and assuming such bounds is very realistic. In other words, if there are agents that do not perform update/move for an unbounded amount of time or do not perform position sensing of their neighbors, they are not effectively part of the swarm. Consider again a networked system of agents that have continuous-time vehicle dynamics and operate on the (infinite) sequence of behaviors *wait–sense–compute–move*. As mentioned before, for such a system, $x^i(t)$ means the next waypoint which is computed during the *compute* state, and no computation is performed during the other states. Similarly, it performs neighbor position/state sensing only during its *sense* state, and no sensing is performed at the other states. For such a system, Assumption 3 means that the sequence *wait–sense–compute–move* is completed in a finite amount of time. Therefore, the low-level control and communication/sensing algorithms should be designed such that this is guaranteed and the agent computes its next waypoint based only on the agents that it has sensed (or communicated with). The systems satisfying Assumption 3 are referred to as the partially asynchronous systems in [19].

Note that, since sets T^i are infinite and there are only a finite number of agents in the swarm, some of them may become neighbors of i only finitely many times, while others become its neighbor infinitely many times as $t \to \infty$. Assumption 3 can be relaxed to state that, given the switching sequence $\sigma(t)$, agent i regularly updates its (perceived) information about members j, which become its neighbors, i.e., $j \in S_i(t)$, infinitely often as time goes to infinity (and therefore its state affects the update in (1) as time goes to infinity).

Under Assumption 3, using the result for the synchronous case and a result from [19], one can show that, as $t \to \infty$, we have $x(t) \to Y_c$ for some Y_c, implying that (3) is satisfied.

Theorem 1: For the N-member swarm in (1), if Assumptions 1 and 3 hold, then, (3) is satisfied as $t \to \infty$, i.e., the swarm member states will asymptotically converge to a common value x^c for some $x^c \in Y(0) = X(0)$.

Proof: The result directly follows from the asynchronous convergence theorem in [19]. For convenience, we will adapt and present the proof here. It is based on induction. Let us denote the concatenation of the states of all the agents as $x(t) = (x^1(t), x^2(t), \ldots, x^N(t)) \in \mathbb{R}^{N \times n}$. Initially, at $t = 0$, we have $x(0) \in \bar{Y}(0)$ by hypothesis. Given that $x(t) \in \bar{Y}(k)$ for some t_k and for all $t \geq t_k$, we will show that there exist a time t_{k+1} such that $x(t) \in \bar{Y}(k+1)$ for all $t \geq t_{k+1}$.

Then, in the light of the discussion for the synchronous case in the preceding section, we have the result. Let

$$x_{pi}(t) = \left(x^1\left(\tau_1^i(t)\right), x^2\left(\tau_2^i(t)\right), \ldots, x^N\left(\tau_N^i(t)\right)\right)$$

denote the "perceived" system state by agent i. We use this notation for convenience. Here, if $j \in S_i(t)$, then $x^j(\tau_j^i(t))$ is the perceived state of neighbor j; otherwise, if $j \notin S_i(t)$, we take $x^j(\tau_j^i(t)) = x^j(t)$ since it does not affect the state update of agent i in (1).

By Assumption 3, since, for every i and $j \in S_i(t)$, we have $t - B < \tau_j^i(t) \leq t$, it is guaranteed that, after time $\bar{t}_k = t_k + B$, we have $\tau_j^i(t) \geq t_k$ for all $t \geq \bar{t}_k$ for all agents i and for all $j \in S_i(t)$. In other words, all agents perform sensing of (or communication with) all of its neighbors (arising due to the switching sequence $\sigma(t)$) by time $\bar{t}_k$ or in, at most, B steps after time t_k—this is guaranteed by Assumption 3. Note that the perceived minimum by individual i

$$m_{pi}(t) = \min_{i=1,\ldots,N} \left\{x^j\left(\tau_j^i(t)\right)\right\} \leq m(t)$$

cannot decrease and the perceived maximum

$$M_{pi}(t) = \max_{i=1,\ldots,N} \left\{x^j\left(\tau_j^i(t)\right)\right\} \geq M(t)$$

cannot increase. The inequalities $m_{pi}(t) \leq m(t)$ and $M_{pi}(t) \geq M(t)$ hold due to the facts that $m(t)$ is nondecreasing ($m(t-1) \leq m(t)$ for all t) and $M(t)$ is nonincreasing ($M(t-1) \geq M(t)$ for all t), and the facts that the perceived $m_{pi}(t)$ and $M_{pi}(t)$ correspond to older information. Therefore, since $x(t) \in \bar{Y}(k)$ for $t \geq t_k$ and $\tau_j^i(t) \geq t_k$ for $t \geq \bar{t}_k = t_k + B$ and for all $j \in S_i(t)$, we have $m_{pi}(t) \geq m(t_k)$ and $M_{pi}(t) \leq M(t_k)$ for all $t \geq \bar{t}_k$, implying that

$$x_{pi}(t) \in \bar{Y}(k) \qquad \forall t \geq \bar{t}_k.$$

In other words, for $t \geq \bar{t}_k$, the agents "perceive" that the state of the system belongs to $\bar{Y}(k)$. Therefore, for all $t \geq \bar{t}_k$, we have $x_{pi}(t) \in \bar{Y}(k)$ for all i. Define $t_{k+1} = \bar{t}_k + INB = t_k + (1 + IN)B$, where I is the length of the uniform connectivity interval in Assumption 1, N is the number of individuals (which is greater than the length of the maximal path in the communication graph), and B is the bound in Assumption 3. Then, from the motion dynamics in (1) and the fact that $x_{pi}(t) \in \bar{Y}(k)$ for all i and for all $t \geq \bar{t}_k$ in light of the discussion about the synchronous case in the preceding section, it is guaranteed that $x^i(t) \in Y(k)$ for all $t \geq t_{k+1}$, implying that

$$x(t) \in \bar{Y}(k+1)$$

for all $t \geq t_{k+1}$, which completes the proof. ■

This result is important because it states that the stability of the system will be preserved (i.e., all the agent states will converge to the same value), even though we have an asynchronous state update mechanism and imperfect information due to time delays on top of the time-varying or switching communication topology. The main arguments of the proof are based on a convexity-type condition and the contraction properties of the iteration (due to the averaging in the agent motion dynamics). The speed of convergence is bounded below by a constant, which depends on the value of $\mu = (1 + IN)B$ since it is guaranteed that contraction will occur in, at most, μ time steps.

A special case of this result occurs when the communication topology is fixed. If this is the case, Assumption 1 basically becomes the following:

Assumption 4: The interaction graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ has a spanning tree.

The corresponding result can be stated as follows:

Corollary 1: For the N-member swarm in (1), if the interaction graph is fixed and if Assumptions 3 and 4 hold, then (3) is satisfied as $t \to \infty$, i.e., the swarm member states will asymptotically converge to a common value x^c for some $x^c \in Y(0) = X(0)$.

V. Extension to General Nonlinear Dynamics

The asynchronous convergence theorem in [19], on which the results of this correspondence are based, is a general result that is not limited to linear iterations and can be applied to nonlinear iterations as well. In fact, as previously mentioned, the two main arguments of the proof of convergence in the linear case are a convexity-type condition and the contraction properties of the iteration. Therefore, as long as these properties are preserved, the results will still hold. Consider the asynchronous version of the nonlinear system considered in [2]. In other words, consider the nonlinear system in which individual i moves according to

$$x^i(t+1) = f_i\left(t, x^1\left(\tau_1^i(t)\right), \ldots, x^N\left(\tau_N^i(t)\right)\right) \qquad \forall t \in T^i \quad (5)$$

and it is stationary otherwise. This iteration can also be represented by $f_i(t, x_{pi}(t))$, where $x_{pi}(t) \in \bar{X} \subset \mathbb{R}^{N \times n}$ is the perceived state of the system by agent i at time t, and the time dependence in f_i is due to the time-dependent communication topology. For simplicity, it is assumed that $f\colon \mathbb{N} \times \bar{X} \to \bar{X}$ is continuous. Here, $\bar{X} = X^N$, with $X \subset \mathbb{R}^n$. In addition, assume that Assumption 1 is satisfied. Furthermore, let the following convexity assumption from [2] be satisfied:

Assumption 5: Associated to each directed graph $(\mathcal{N}, \mathcal{A})$ with node set $\mathcal{N} = \{1, \ldots, N\}$, each agent i, and each state $x \in \bar{X}$ there is a compact set $e_i(\mathcal{A})(x) \subset X$ that satisfies four conditions:

1) $f_i(t, x) \in e_i(\mathcal{A}(t))(x), \forall t \in T; \forall x \in \bar{X}$.
2) $e_i(\mathcal{A}(t))(x^1, \ldots, x^N) = \{x^i\}$ whenever the states of agent i and agents $j \in S_i(t)$ are equal.
3) $e_i(\mathcal{A}(t))(x^1, \ldots, x^N)$ is contained in the relative interior of the convex hull of the states of agent i and agents $j \in S_i(t)$ whenever the states of agent i and agents $j \in S_i(t)$ are not equal.
4) $e_i(\mathcal{A})(x)$ depends continuously on x.

Under this assumption, Moreau [2] showed that the synchronous version of the nonlinear system (5) will result in the convergence of the states of all the agents to the same value. Assumption 5, together with Assumptions 1 and 3 also guarantee that the asynchronous system in (5) will converge as well. To this end, define the convex hull of the agent states as $X(t)$. In other words, define

$$X(t) = conv\{x^1(t), \ldots, x^N(t)\}$$

and $Y(k)$ and $\bar{Y}(k)$ as before. From the results in [2], we know that, under Assumption 1, for the synchronous case, there is a sequence of sets $\bar{Y}(k)$ satisfying $Y_c \subset \cdots \subset \bar{Y}(k+1) \subset \bar{Y}(k) \subset \cdots \subset \bar{Y}(0) \subset \mathbb{R}^{N \times n}$, and any sequence $\{y_k \in \bar{Y}(k)\}$ converges to Y_c. Using arguments that are similar to those in Theorem 1, one can prove the following result for the nonlinear system in (5):

Theorem 2: For the N-member swarm in (5), if Assumptions 1 and 3 hold, together with Assumption 5, then, (3) is satisfied as $t \to \infty$, i.e., the swarm member states will asymptotically converge to a common value x^c for some $x^c \in Y(0) = X(0)$.

VI. Simulation Results

In this section, we provide numerical simulation examples. We chose $n = 3$, (i.e., $\mathbb{R}^3$). To achieve asynchronism at each time step, the swarm members are set up to sense their neighbor states and to

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TABLE I
PSEUDOCODE

```
initialize p̄_sense and p̄_move
initialize the positions of the agents (randomly)
determine the neighborhoods S_i, i = 1, ..., N (fixed topology only)
for t=1:final_time do
  for each agent i do
    determine the set of its neighbors S_i(t) (dynamic topology only)
    for each agent j ∈ S_i(t)
      generate p^{ij}_sense(t)
      if p^{ij}_sense(t) > p̄_sense
        obtain the state information of agent j
      end
    end
    generate p^i_move(t)
    if p^i_move(t) > p̄_move,
      update state using (1)
    else
      keep current state using (2)
    end
  end
end
```

TABLE II
RANDOM NEIGHBORHOOD

```
initialize p̄_nbr
for each agent i do
  for each agent j ≠ i do
    generate p^{ij}_nbr(t)
    if p^i_nbr(t) > p̄_nbr
      j ∈ S_i(t)
    end
  end
end
```

update their own state with some probability. In particular, we defined two threshold probabilities $0 < \bar{p}_{\text{sense}} < 1$ and $0 < \bar{p}_{\text{move}} < 1$. At each time instant t, for each individual i, a total of $(N_i(t) + 1)$ random numbers, which include $p^{ij}_{\text{sense}}(t)$, $j = 1, \ldots, N_i(t)$, and one $p^i_{\text{move}}(t)$, are generated with uniform probability density in the interval [0, 1]. Then, if $p^{ij}_{\text{sense}}(t) > \bar{p}_{\text{sense}}$, then member i receives the current state of its neighbor $j \in S_i(t)$. Otherwise, it keeps the old state information of agent j. Similarly, if, at step t, we have $p^i_{\text{move}}(t) > \bar{p}_{\text{move}}$, then individual i updates its state according to (1). Otherwise, it keeps its current state according to (2). In other words, the agents move based on the pseudocode shown in Table I. Note that this implementation is not a real discrete-event-based asynchronous system. Instead, it mimics such systems and is sufficient for illustrating/verifying the theoretical results obtained in this correspondence. There could be various procedures for determining the neighbors of the agents both in the fixed and dynamic topologies. One procedure is to assign the neighbors randomly, and this is exactly the procedure that we used here. The pseudocode for that is shown in Table II. In particular, a total of $N \times (N - 1)$ random numbers were generated before the main loop for the fixed topology case and at each time step t for the dynamic topology case (i.e., different random numbers $p^{ij}_{\text{nbr}}(t)$ for every possible pair (i, j), $j \neq i$). In addition, we defined $\bar{p}_{\text{nbr}}$, and if, at time t, for a pair (i, j), $j \neq i$, we have $p^{ij}_{\text{nbr}}(t) > \bar{p}_{\text{nbr}}$, then agent j is assigned as a neighbor of agent i for time t, i.e., $j \in S_i(t)$. Since $p^{ij}_{\text{nbr}}(t)$ is independent and possibly different from $p^{ji}_{\text{nbr}}(t)$, $j \in S_i(t)$ does not imply $i \in S_j(t)$ or vice versa.

Fig. 1 shows a simulation of a system with fixed communication topology for $N = 100$ members. We chose the initial states of the agents randomly in the interval [0, 1]. Moreover, we assigned the (fixed) neighbors of the agents also randomly with a probability of 0.1 (i.e., $\bar{p}_{\text{nbr}} = 0.9$). In addition, for these simulations, we used $\bar{p}_{\text{sense}} = 0.5$ and $\bar{p}_{\text{move}} = 0.5$. As seen from the figure, all the agent states converge to the same value. The three distinct lines in the figure are due to the fact that the agent states evolve in $\mathbb{R}^3$ (i.e., $x^i \in \mathbb{R}^3$), and all three coordinates are plotted on the same plot. Fig. 2 shows the plot of the sum of the distances between the states of the agents

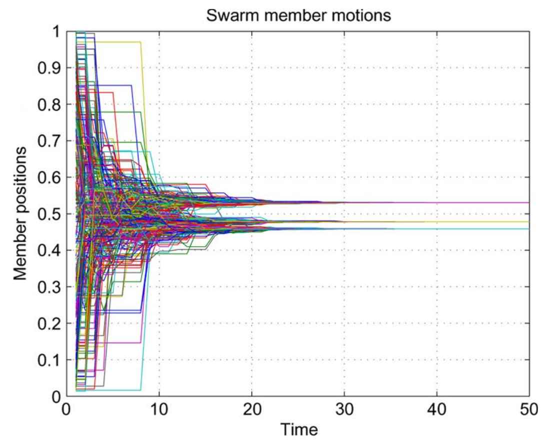

Fig. 1. States of the agents (fixed topology).

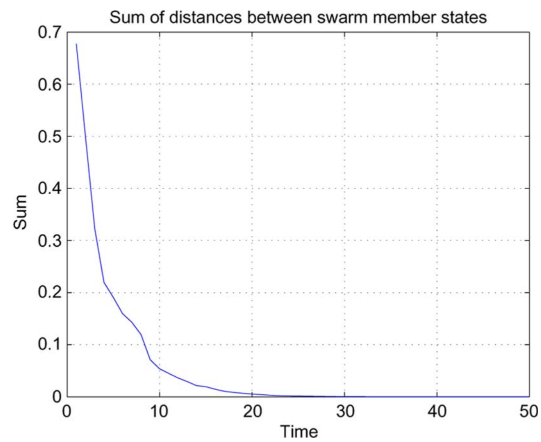

Fig. 2. Distance between agent states (fixed topology).

$$e(t) = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left\| x^i(t) - x^j(t) \right\|$$

with respect to time. As predicted by the analysis, it is seen to converge to zero. Other fixed neighborhood topologies that we experimented with are the fully connected and the cyclic (ring) neighborhood topologies (results not shown here). The fully connected neighborhood topology converges very fast. In the ring topology, on the other hand, the convergence is much slower since the length of the path in the spanning tree there is much longer. The only point here is that the spanning tree assumption should be satisfied. For the fully connected and ring topologies, it is always satisfied; however, for the random neighborhood, for large values of $\bar{p}_{\text{nbr}}$, it might not be satisfied, and for these cases, convergence is not guaranteed.

For the dynamic topology case, in addition to the asynchronism and time delays at each time step, the neighbors of the agents were randomly reassigned based on the procedure shown in Table II. Figs. 3 and 4 show the plots of the agent states and the sum of the distances between them, respectively. For these simulations, we again used the same probability values $\bar{p}_{\text{sense}} = 0.5$, $\bar{p}_{\text{move}} = 0.5$, and $\bar{p}_{\text{nbr}} = 0.9$. As one can see, once more, the states of all the agents converge

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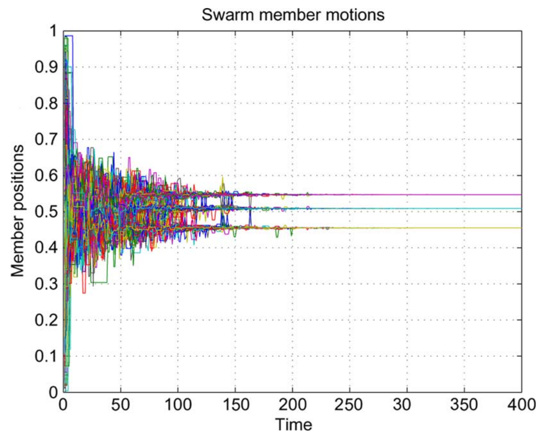

Fig. 3. States of the agents (dynamic topology).

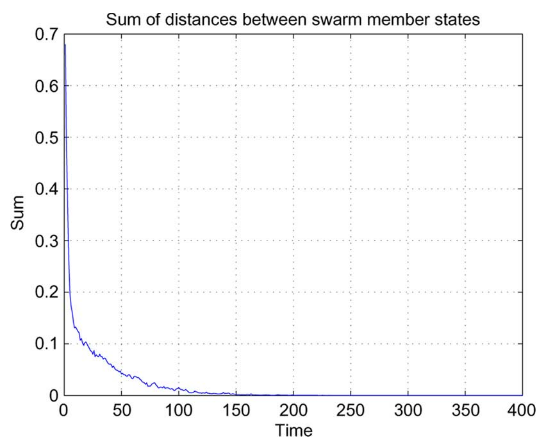

Fig. 4. Distance between agent states (dynamic topology).

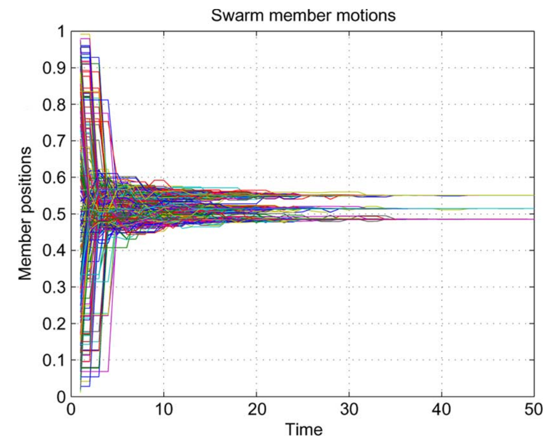

Fig. 5. States of agents ($\bar{p}_{\rm nbr} = 0.5$, $\bar{p}_{\rm sense} = 0.5$, and $\bar{p}_{\rm move} = 0.5$).

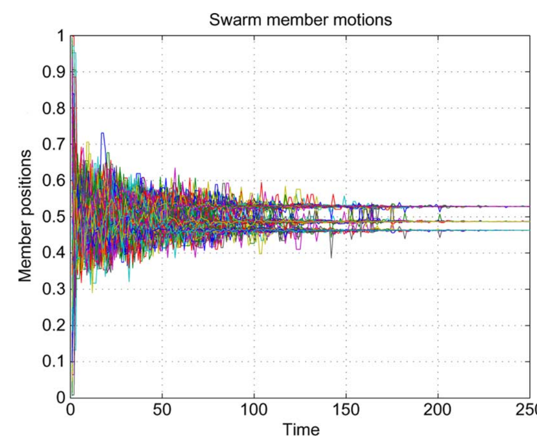

Fig. 6. States of agents ($\bar{p}_{\rm nbr} = 0.9$, $\bar{p}_{\rm sense} = 0.5$, and $\bar{p}_{\rm move} = 0.2$).

to a common value. For this case, however, convergence is slower compared to the corresponding fixed topology case.

Although we did not explicitly impose a bound B on the time delay and the time between two subsequent moves (recall Assumption 3), effectively such a bound exists in the preceding simulations despite the fact that, with the preceding type of implementation, theoretically infinite delays are also possible. In particular, in the simulations, by choosing the values of $\bar{p}_{\rm sense}$, $\bar{p}_{\rm move}$, and $\bar{p}_{\rm nbr}$, one can change the speed of convergence (of the implemented simulation algorithm). In fact, decreasing $\bar{p}_{\rm sense}$, $\bar{p}_{\rm move}$, or $\bar{p}_{\rm nbr}$ leads to faster convergence (since it results in higher probabilities to sense, move, and become neighbors), whereas increasing $\bar{p}_{\rm sense}$, $\bar{p}_{\rm move}$, or $\bar{p}_{\rm nbr}$ leads to slower convergence. This is because decreasing $\bar{p}_{\rm sense}$ and $\bar{p}_{\rm move}$ implies that the agents will move and sense more often (implying that, effectively, bound B in Assumption 3 will decrease). In other words, the values of these parameters determine the resulting effective value of bound B, which, on the other hand, affects the speed of convergence. Decreasing $\bar{p}_{\rm nbr}$, on the other hand, leads to a more connected communication/interaction topology (implying that, effectively, bound I in Assumption 1 will decrease). Note from the proof of Theorem 1 that both B and I affect the worst case time for the set to contract. The plots in Figs. 5–7 investigate these effects. For the simulation in Fig. 5, the probability threshold for the agents becoming neighbors was decreased from $\bar{p}_{\rm nbr} = 0.9$ to $\bar{p}_{\rm nbr} = 0.5$ (while keeping $\bar{p}_{\rm sense} = 0.5$ and $\bar{p}_{\rm move} = 0.5$). As one can see from the figure, the system converges (i.e., consensus is achieved) much faster, as expected. For the simulation in Fig. 6, the probability threshold for the agents to move was decreased to $\bar{p}_{\rm move} = 0.2$ (while keeping $\bar{p}_{\rm sense} = 0.5$ and $\bar{p}_{\rm nbr} = 0.9$). This resulted in only very slight difference, compared to the simulation in Fig. 3, and substantial increase in convergence speed is not achieved. This is probably because, even though the agents move more often, they still use the same amount of outdated information, which prevents the increase in the convergence speed. For the simulation in Fig. 7, the probability threshold for the agents to sense was decreased to $\bar{p}_{\rm sense} = 0.2$ (while keeping $\bar{p}_{\rm move} = 0.5$ and $\bar{p}_{\rm nbr} = 0.9$). This also resulted in a faster convergence compared to the case in Fig. 3. Note that the synchronous case considered in the literature corresponds to the case with $\bar{p}_{\rm move} = 0$ and $\bar{p}_{\rm sense} = 0$ (move at each step with always the current information), which converges much faster, compared to the corresponding asynchronous case (for both the fixed and dynamic topologies) with time delays considered here. Note also that the speed of convergence is not affected by the dimension of the state space n (since actually each dimension is independent and they do not affect each other). In contrast, the number of agents can affect the convergence rate, as expected from the bound obtained in the proof of Theorem 1. In particular, a higher number N of agents may result in slower convergence.

In addition, in the probabilistic neighborhood (that we used for illustration here), there are other possible methods for defining the dynamic neighborhood topology. One such method is the nearest neighbor rule

$$S_i(t) = \left\{ j | j \neq i, \|x^i(t) - x^j(t)\| \leq \delta \right\}.$$

We experimented with that neighborhood as well (results not shown here). We would like to emphasize that the objective here is not to

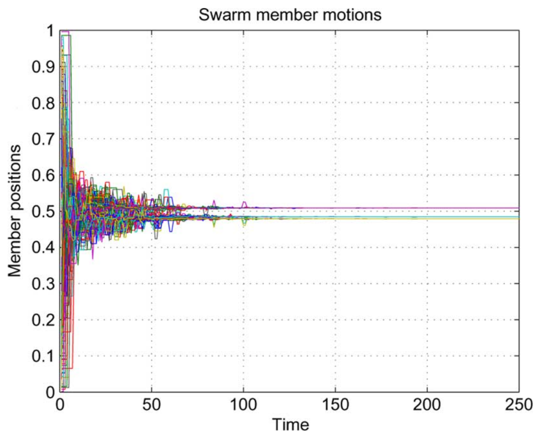

Fig. 7. States of agents ($\bar{p}_{\rm nbr} = 0.9$, $\bar{p}_{\rm sense} = 0.2$, and $\bar{p}_{\rm move} = 0.5$).

perform a comprehensive simulation study for many different cases; instead, it is to illustrate the theoretical results obtained in the preceding sections. In addition, all the simulations that we performed for all the cases (including those not shown) support the theoretical results obtained in this correspondence.

VII. Concluding Remarks

In this correspondence, we present an n-dimensional discrete-time asynchronous swarm model, which can also include sensing or communication delays. We show under the assumption that old information is uniformly purged from the system that asymptotic convergence of the states of all the agents to the same value is achieved despite the presence of delays and asynchronism, thus extending some earlier results that appeared in the literature. The discussed model appears to be more natural or suitable for multiagent systems since it does not require global clock for synchronization. Future research could focus on analyzing the stability and performance of the system under sensing errors and uncertainties in addition to the asynchronism and time delays.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[2] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[3] L. Moreau, "A note on leaderless coordination via bidirectional and unidirectional time-dependent communication," in *Proc. Int. Symp. Math. Theory Netw. Syst.*, Leuven, Belgium, Jul. 5–9, 2004.

[4] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[5] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 121–127, Jan. 2005.

[6] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[7] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[8] R. Sepulchre, D. Palay, and N. E. Leonard, "Collective motion and oscillator synchronization," in *Proc. Block Island Workshop Cooperative Control*, V. J. Kumar, N. E. Leonard, and A. S. Morse, Eds. London, U.K.: Springer-Verlag, 2005, vol. 309, pp. 189–205.

[9] Y. Liu, K. M. Passino, and M. M. Polycarpou, "Stability analysis of one-dimensional asynchronous swarms," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1848–1854, Oct. 2003.

[10] Y. Liu, K. M. Passino, and M. M. Polycarpou, "Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology," *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 76–95, Jan. 2003.

[11] V. Gazi and K. M. Passino, "Stability of a one-dimensional discrete-time asynchronous swarm," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 35, no. 4, pp. 834–841, Aug. 2005.

[12] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robot. Autom.*, vol. 20, no. 2, pp. 243–255, Apr. 2004.

[13] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem—The asynchronous case," in *Proc. Conf. Decision Control*, Paradise Island, Bahamas, Dec. 2004, pp. 1926–1931.

[14] N. Gordon, I. A. Wagner, and A. M. Bruckstein, "Gathering multiple robotic a(ge)nts with limited sensing capabilities," in *Proc. 4th Int. Workshop ANTS, Brussels, Belgium*, M. Dorigo, M. Birattari, C. Blum, L. M. Gambardella, F. Mondada, and T. Stützle, Eds. Berlin, Germany: Springer-Verlag, Sep. 5–8, 2004, vol. 3172, pp. 142–153.

[15] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of autonomous mobile robots with limited visibility," in *Proc. 18th Int. STACS, Dresden, Germany*. New York: Springer-Verlag, Feb. 15–17, 2001, vol. 2010, pp. 247–258.

[16] G. Beni, "Order by disordered action in swarms," in *Proc. SAB Int. Workshop Swarm Robot.*, Lecture Notes in Computer Science, E. Sahin and W. M. Spears Eds. New York: Springer-Verlag, Jul. 2004, vol. 3342, pp. 153–171.

[17] A. T. Şamiloglu, V. Gazi, and A. B. Koku, "Effects of asynchronism and neighborhood size on clustering in self-propelled particle systems," in *Proc. ISCIS*, A. Levi *et al.*, Ed. New York: Springer-Verlag, Nov. 2006, vol. 4263, pp. 665–676.

[18] A. T. Şamiloglu, V. Gazi, and A. B. Koku, "Asynchronous cyclic pursuit," in *Proc. 9th Conf. SAB*, S. Nolfi *et al.*, Ed. New York: Springer-Verlag, Sep. 2006, vol. 4095, pp. 667–678.

[19] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Belmont, MA: Athena Scientific, 1997.

[20] D. Angeli and P.-A. Bliman, *Stability of Leaderless Multi-Agent Systems. Extension of a Result by Moreau*, Nov. 2004. arXiv:math.OC/0411338 v1.

[21] L. Fang, P. J. Antsaklis, and A. Tzimas, "Asynchronous consensus protocols: Preliminary results, simulations and open questions," in *Proc. Conf. Decision Control, Eur. Control Conf.*, Seville, Spain, Dec. 2005, pp. 2194–2199.

[22] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. Ñ. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," in *Proc. Conf. Decision Control, Eur. Control Conf.*, Seville, Spain, Dec. 2005, pp. 2996–3000.

[23] V. Gazi, "Stability of a discrete-time asynchronous swarm with time-dependent communication links," in *Proc. Syst. Control Theory Workshop*, Sep. 2005, pp. 37–46.

[24] R. P. Abelson, E. Aronson, W. J. McGuire, T. M. Newcomb, M. J. Rosenberg and P. H. Tannenbaum, Eds., *Theories of Cognitive Consistency: A Sourcebook*. Chicago, IL: Rand McNally, 1968.

[25] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 1995.